Chapter **10**

VECTOR ALGEBRA

10.1 Overview

10.1.1 A quantity that has magnitude as well as direction is called a vector.

10.1.2 The unit vector in the direction of $\vec{a}$ is given by $\frac{\vec{a}}{|\vec{a}|}$ and is represented by $\hat{a}$.

10.1.3 Position vector of a point P (x, y, z) is given as $\overrightarrow{OP} = x\hat{i} + y\hat{j} + z\hat{k}$ and its magnitude as $|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$, where O is the origin.

10.1.4 The scalar components of a vector are its direction ratios, and represent its projections along the respective axes.

10.1.5 The magnitude r, direction ratios (a, b, c) and direction cosines (l, m, n) of any vector are related as:

$$l = \frac{a}{r},\ m = \frac{b}{r},\ n = \frac{c}{r}.$$

10.1.6 The sum of the vectors representing the three sides of a triangle taken in order is $\vec{0}$

10.1.7 The triangle law of vector addition states that "If two vectors are represented by two sides of a triangle taken in order, then their sum or resultant is given by the third side taken in opposite order".

10.1.8 ***Scalar multiplication***

If $\vec{a}$ is a given vector and λ a scalar, then $\lambda\vec{a}$ is a vector whose magnitude is $|\lambda\vec{a}| = |\lambda|\,|\vec{a}|$. The direction of $\lambda\vec{a}$ is same as that of $\vec{a}$ if λ is positive and, opposite to that of $\vec{a}$ if λ is negative.

10.1.9 *Vector joining two points*

If P_1 (x_1, y_1, z_1) and P_2 (x_2, y_2, z_2) are any two points, then

$$\overrightarrow{P_1P_2} = (x_2 - x_1)\,\hat{i} + (y_2 - y_1)\,\hat{j} + (z_2 - z_1)\,\hat{k}$$

$$|\overrightarrow{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

10.1.10 *Section formula*

The position vector of a point R dividing the line segment joining the points P and Q whose position vectors are $\vec{a}$ and $\vec{b}$

(i) in the ratio $m : n$ internally, is given by $\dfrac{n\vec{a} + m\vec{b}}{m+n}$

(ii) in the ratio $m : n$ externally, is given by $\dfrac{m\vec{b} - n\vec{a}}{m-n}$

10.1.11 Projection of $\vec{a}$ along $\vec{b}$ is $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$ and the Projection vector of $\vec{a}$ along $\vec{b}$ is $\left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\right)\vec{b}$.

10.1.12 *Scalar or dot product*

The scalar or dot product of two given vectors $\vec{a}$ and $\vec{b}$ having an angle θ between them is defined as

$$\vec{a}\,.\,\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$$

10.1.13 *Vector or cross product*

The cross product of two vectors $\vec{a}$ and $\vec{b}$ having angle θ between them is given as $\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}|\sin\theta\;\hat{n}$, where $\hat{n}$ is a unit vector perpendicular to the plane containing $\vec{a}$ and $\vec{b}$ and $\vec{a}, \vec{b}, \hat{n}$ form a right handed system.

10.1.14 If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are two vectors and λ is any scalar, then

$$\vec{a} + \vec{b} = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k}$$

$$\lambda\,\vec{a} = (\lambda\, a_1)\hat{i} + (\lambda\, a_2)\hat{j} + (\lambda\, a_3)\hat{k}$$

$$\vec{a} \cdot \vec{b} = a_1\, b_1 + a_2\, b_2 + a_3\, b_3$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = (b_1c_2 - b_2c_1)\,\hat{i} + (a_2c_1 - c_1c_2)\,\hat{j} + (a_1b_b - a_2b_1)\,\hat{k}$$

Angle between two vectors $\vec{a}$ and $\vec{b}$ is given by

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|} = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}$$

10.2 Solved Examples

Short Answer (S.A.)

Example 1 Find the unit vector in the direction of the sum of the vectors

$\vec{a} = 2\,\hat{i} - \hat{j} + 2\,\hat{k}$ and $\vec{b} = -\hat{i} + \hat{j} + 3\,\hat{k}$.

Solution Let $\vec{c}$ denote the sum of $\vec{a}$ and $\vec{b}$. We have

$$\vec{c} = (2\,\hat{i} - \hat{j} + 2\,\hat{k}) + (-\hat{i} + \hat{j} + 3\,\hat{k}) = \hat{i} + 5\,\hat{k}$$

Now $|\vec{c}| = \sqrt{1^2 + 5^2} = \sqrt{26}$.

Thus, the required unit vector is $\hat{c} = \frac{\vec{c}}{|\vec{c}|} = \frac{1}{\sqrt{26}}(\hat{i} + 5\hat{k}) = \frac{1}{\sqrt{26}}\hat{i} + \frac{5}{\sqrt{26}}\hat{k}$.

Example 2 Find a vector of magnitude 11 in the direction opposite to that of $\overrightarrow{PQ}$, where P and Q are the points (1, 3, 2) and (–1, 0, 8), respetively.

Solution The vector with initial point P (1, 3, 2) and terminal point Q (–1, 0, 8) is given by

$$\overrightarrow{PQ} = (-1-1)\,\hat{i} + (0-3)\,\hat{j} + (8-2)\,\hat{k} = -2\,\hat{i} - 3\,\hat{j} + 6\,\hat{k}$$

Thus $\quad \overrightarrow{QP} = -\overrightarrow{PQ} = 2\hat{i} + 3\hat{j} - 6\hat{k}$

$$\Rightarrow |\overrightarrow{QP}| = \sqrt{2^2 + 3^2 + (-6)^2} = \sqrt{4+9+36} = \sqrt{49} = 7$$

Therefore, unit vector in the direction of $\overrightarrow{QP}$ is given by

$$\widehat{QP} = \frac{\overrightarrow{QP}}{|\overrightarrow{QP}|} = \frac{2\hat{i} + 3\hat{j} - 6\hat{k}}{7}$$

Hence, the required vector of magnitude 11 in direction of $\overrightarrow{QP}$ is

$$11\ \widehat{QP} = 11\left(\frac{2\hat{i} + 3\hat{j} - 6\hat{k}}{7}\right) = \frac{22}{7}\hat{i} + \frac{33}{7}\hat{j} - \frac{66}{7}\hat{k}.$$

Example 3 Find the position vector of a point R which divides the line joining the two points P and Q with position vectors $\overrightarrow{OP} = 2\vec{a} + \vec{b}$ and $\overrightarrow{OQ} = \vec{a} - 2\vec{b}$, respectively, in the ratio 1:2, (i) internally and (ii) externally.

Solution (i) The position vector of the point R dividing the join of P and Q internally in the ratio 1:2 is given by

$$\overrightarrow{OR} = \frac{2(2\vec{a} + \vec{b}) + 1(\vec{a} - 2\vec{b})}{1+2} = \frac{5\vec{a}}{3}.$$

(ii) The position vector of the point R′ dividing the join of P and Q in the ratio 1 : 2 externally is given by

$$\overrightarrow{\mathrm{OR}'} = \frac{2(2\vec{a}+\vec{b})-1(\vec{a}-2\vec{b})}{2-1} = 3\vec{a}+4\vec{b}\,.$$

Example 4 If the points (–1, –1, 2), (2, m, 5) and (3,11, 6) are collinear, find the value of m.

Solution Let the given points be A (–1, –1, 2), B (2, m, 5) and C (3, 11, 6). Then

$$\overrightarrow{\mathrm{AB}} = (2+1)\,\hat{i} + (m+1)\,\hat{j} + (5-2)\,\hat{k} = 3\,\hat{i} + (m+1)\,\hat{j} + 3\hat{k}$$

and
$$\overrightarrow{\mathrm{AC}} = (3+1)\,\hat{i} + (11+1)\,\hat{j} + (6-2)\,\hat{k} = 4\,\hat{i} + 12\,\hat{j} + 4\hat{k}\,.$$

Since A, B, C, are collinear, we have $\overrightarrow{\mathrm{AB}} = \lambda\,\overrightarrow{\mathrm{AC}}$, i.e.,

$$(3\,\hat{i} + (m+1)\,\hat{j} + 3\hat{k}) = \lambda(4\hat{i} + 12\hat{j} + 4\hat{k})$$

$\Rightarrow$ $3 = 4\lambda$ and $m + 1 = 12\lambda$

Therefore $m = 8$.

Example 5 Find a vector $\vec{r}$ of magnitude $3\sqrt{2}$ units which makes an angle of $\frac{\pi}{4}$ and $\frac{\pi}{2}$ with y and z - axes, respectively.

Solution Here $m = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ and $n = \cos\frac{\pi}{2} = 0$.

Therefore, $l^2 + m^2 + n^2 = 1$ gives

$$l^2 + \frac{1}{2} + 0 = 1$$

$\Rightarrow$ $l = \pm\frac{1}{\sqrt{2}}$

Hence, the required vector $\vec{r} = 3\sqrt{2}\ (l\hat{i} + m\hat{j} + n\hat{k})$ is given by

$$\vec{r} = 3\sqrt{2}\ (\pm\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j} + 0\hat{k}) = \vec{r} = \pm 3\hat{i} + 3\hat{j}\,.$$

Example 6 If $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{b} = \hat{i} + \hat{j} - 2\hat{k}$ and $\vec{c} = \hat{i} + 3\hat{j} - \hat{k}$, find λ such that $\vec{a}$ is perpendicular to $\lambda\vec{b} + \vec{c}$.

Solution We have

$$\lambda\vec{b} + \vec{c} = \lambda\ (\hat{i} + \hat{j} - 2\hat{k}) + (\hat{i} + 3\hat{j} - \hat{k})$$

$$= (\lambda + 1)\ \hat{i} + (\lambda + 3)\hat{j} - (2\lambda + 1)\ \hat{k}$$

Since $\vec{a} \perp (\lambda\vec{b} + \vec{c})$, $\vec{a} \cdot (\lambda\vec{b} + \vec{c}) = 0$

$$\Rightarrow (2\hat{i} - \hat{j} + \hat{k})\,.\,[(\lambda + 1)\ \hat{i} + (\lambda + 3)\ \hat{j} - (2\lambda + 1)\hat{k}\,] = 0$$

$$\Rightarrow 2\,(\lambda + 1) - (\lambda + 3) - (2\lambda + 1) = 0$$

$$\Rightarrow \lambda = -\,2.$$

Example 7 Find all vectors of magnitude $10\sqrt{3}$ that are perpendicular to the plane of $\hat{i} + 2\hat{j} + \hat{k}$ and $-\hat{i} + 3\hat{j} + 4\hat{k}$.

Solution Let $\vec{a} = \hat{i} + 2\hat{j} + \hat{k}$ and $\vec{b} = -\hat{i} + 3\hat{j} + 4\hat{k}$. Then

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 1 \\ -1 & 3 & 4 \end{vmatrix} = \hat{i}(8-3) - \hat{j}(4+1) + \hat{k}(3+2) = 5\hat{i} - 5\hat{j} + 5\hat{k}$$

$$\Rightarrow \qquad \left|\vec{a} \times \vec{b}\right| = \sqrt{(5)^2 + (-5)^2 + (5)^2} = \sqrt{3(5)^2} = 5\sqrt{3}\,.$$

Therefore, unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$ is given by

$$\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}=\frac{5\hat{i}-5\hat{j}+5\hat{k}}{5\sqrt{3}}$$

Hence, vectors of magnitude of $10\sqrt{3}$ that are perpendicular to plane of $\vec{a}$ and $\vec{b}$

are $\pm 10\sqrt{3}\left(\frac{5\hat{i}-5\hat{j}+5\hat{k}}{5\sqrt{3}}\right)$, i.e., $\pm 10(\hat{i}-\hat{j}+\hat{k})$.

Long Answer (L.A.)

Example 8 Using vectors, prove that cos (A – B) = cosA cosB + sinA sinB.

Solution Let $\widehat{OP}$ and $\widehat{OQ}$ be unit vectors making angles A and B, respectively, with positive direction of x-axis. Then $\angle$QOP = A – B [Fig. 10.1]

We know $\widehat{OP} = \overrightarrow{OM} + \overrightarrow{MP} = \hat{i}\cos A + \hat{j}\sin A$ and $\widehat{OQ} = \overrightarrow{ON} + \overrightarrow{NQ} = \hat{i}\cos B + \hat{j}\sin B$.

By definition $\widehat{OP}\,.\,\widehat{OQ} = |\widehat{OP}|\,|\widehat{OQ}|\cos(\text{A-B})$

$$= \cos(A - B) \qquad \text{... (1)} \qquad \left(\because |\widehat{OP}| = 1 = |\widehat{OQ}|\right)$$

In terms of components, we have

$\widehat{OP}\,.\,\widehat{OQ} = (\hat{i}\cos A + \hat{j}\sin A).(\hat{i}\cos B + \hat{j}\sin B)$

= cosA cosB + sinA sinB ... (2)

From (1) and (2), we get

cos (A – B) = cosA cosB + sinA sinB.

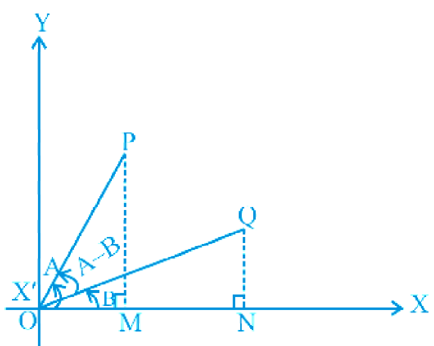

Fig. 10.1

Example 9 Prove that in a Δ ABC, $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$, where *a*, *b*, *c* represent the magnitudes of the sides opposite to vertices A, B, C, respectively.

Solution Let the three sides of the triangle BC, CA and AB be represented by $\vec{a}, \vec{b}$ and $\vec{c}$, respectively [Fig. 10.2].

We have $\vec{a} + \vec{b} + \vec{c} = \vec{0}$. i.e., $\vec{a} + \vec{b} = -\vec{c}$

which pre cross multiplying by $\vec{a}$, and post cross multiplying by $\vec{b}$, gives

$$\vec{a} \times \vec{b} = \vec{c} \times \vec{a}$$

and $$\vec{a} \times \vec{b} = \vec{b} \times \vec{c}$$

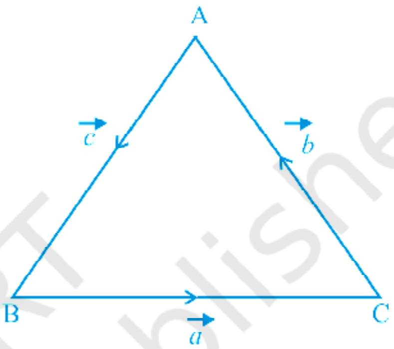

Fig. 10.2

respectively. Therefore,

$$\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$$

$$\Rightarrow \quad |\vec{a} \times \vec{b}| = |\vec{b} \times \vec{c}| = |\vec{c} \times \vec{a}|$$

$$\Rightarrow \quad |\vec{a}||\vec{b}| \sin(\pi - C) = |\vec{b}||\vec{c}| \sin(\pi - A) = |\vec{c}||\vec{a}| \sin(\pi - B)$$

$$\Rightarrow \quad ab \sin C = bc \sin A = ca \sin B$$

Dividing by *abc*, we get

$$\frac{\sin C}{c} = \frac{\sin A}{a} = \frac{\sin B}{b} \quad \text{i.e.} \quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Objective Type Questions

Choose the correct answer from the given four options in each of the Examples 10 to 21.

Example 10 The magnitude of the vector $6\hat{i} + 2\hat{j} + 3\hat{k}$ is

(A) 5 (B) 7 (C) 12 (D) 1

Solution (B) is the correct answer.

Example 11 The position vector of the point which divides the join of points with position vectors $\vec{a}+\vec{b}$ and $2\vec{a}-\vec{b}$ in the ratio 1 : 2 is

(A) $\frac{3\vec{a}+2\vec{b}}{3}$ (B) $\vec{a}$ (C) $\frac{5\vec{a}-\vec{b}}{3}$ (D) $\frac{4\vec{a}+\vec{b}}{3}$

Solution (D) is the correct answer. Applying section formula the position vector of the required point is

$$\frac{2(\vec{a}+\vec{b})+1(2\vec{a}-\vec{b})}{2+1}=\frac{4\vec{a}+\vec{b}}{3}$$

Example 12 The vector with initial point P (2, –3, 5) and terminal point Q(3, –4, 7) is

(A) $\hat{i}-\hat{j}+2\hat{k}$ (B) $5\hat{i}-7\hat{j}+12\hat{k}$

(C) $-\hat{i}+\hat{j}-2\hat{k}$ (D) None of these

Solution (A) is the correct answer.

Example 13 The angle between the vectors $\hat{i}-\hat{j}$ and $\hat{j}-\hat{k}$ is

(A) $\frac{\pi}{3}$ (B) $\frac{2\pi}{3}$ (C) $\frac{-\pi}{3}$ (D) $\frac{5\pi}{6}$

Solution (B) is the correct answer. Apply the formula $\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}|.|\vec{b}|}$.

Example 14 The value of λ for which the two vectors $2\hat{i}-\hat{j}+2\hat{k}$ and $3\hat{i}+\lambda\hat{j}+\hat{k}$ are perpendicular is

(A) 2 (B) 4 (C) 6 (D) 8

Solution (D) is the correct answer.

Example 15 The area of the parallelogram whose adjacent sides are $\hat{i}+\hat{k}$ and $2\hat{i}+\hat{j}+\hat{k}$ is

(A) $\sqrt{2}$ (B) $\sqrt{3}$ (C) 3 (D) 4

Solution (B) is the correct answer. Area of the parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b}$ is $|\vec{a}\times\vec{b}|$.

Example 16 If $|\vec{a}| = 8$, $|\vec{b}| = 3$ and $|\vec{a}\times\vec{b}| = 12$, then value of $\vec{a}.\vec{b}$ is

(A) $6\sqrt{3}$ (B) $8\sqrt{3}$ (C) $12\sqrt{3}$ (D) None of these

Solution (C) is the correct answer. Using the formula $|\vec{a}\times\vec{b}| = |\vec{a}|.|\vec{b}|\,|\sin\theta|$, we get

$\theta = \pm\frac{\pi}{6}$.

Therefore, $\vec{a}.\vec{b} = |\vec{a}|.|\vec{b}|\cos\theta = 8 \times 3 \times \frac{\sqrt{3}}{2} = 12\sqrt{3}$.

Example 17 The 2 vectors $\hat{j}+\hat{k}$ and $3\hat{i}-\hat{j}+4\hat{k}$ represents the two sides AB and AC, respectively of a ΔABC. The length of the median through A is

(A) $\frac{\sqrt{34}}{2}$ (B) $\frac{\sqrt{48}}{2}$ (C) $\sqrt{18}$ (D) None of these

Solution (A) is the correct answer. Median $\overrightarrow{AD}$ is given by

$$|\overrightarrow{AD}| = \frac{1}{2}|3\hat{i}+\hat{j}+5\hat{k}| = \frac{\sqrt{34}}{2}$$

Example 18 The projection of vector $\vec{a}=2\hat{i}-\hat{j}+\hat{k}$ along $\vec{b}=\hat{i}+2\hat{j}+2\hat{k}$ is

(A) $\frac{2}{3}$ (B) $\frac{1}{3}$ (C) 2 (D) $\sqrt{6}$

Solution (A) is the correct answer. Projection of a vector $\vec{a}$ on $\vec{b}$ is

$$\frac{\vec{a}.\vec{b}}{|\vec{b}|} = \frac{(2\hat{i} - \hat{j} + \hat{k}).(\hat{i} + 2\hat{j} + 2\hat{k})}{\sqrt{1+4+4}} = \frac{2}{3}.$$

Example 19 If $\vec{a}$ and $\vec{b}$ are unit vectors, then what is the angle between $\vec{a}$ and $\vec{b}$ for $\sqrt{3}\vec{a} - \vec{b}$ to be a unit vector?

(A) 30° (B) 45° (C) 60° (D) 90°

Solution (A) is the correct answer. We have

$$(\sqrt{3}\vec{a} - \vec{b})^2 = 3\vec{a}^2 + \vec{b}^2 - 2\sqrt{3}\,\vec{a}.\vec{b}$$

$$\Rightarrow \quad \vec{a}.\vec{b} = \frac{\sqrt{3}}{2} \Rightarrow \cos\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = 30°.$$

Example 20 The unit vector perpendicular to the vectors $\hat{i} - \hat{j}$ and $\hat{i} + \hat{j}$ forming a right handed system is

(A) $\hat{k}$ (B) $-\hat{k}$ (C) $\frac{\hat{i} - \hat{j}}{\sqrt{2}}$ (D) $\frac{\hat{i} + \hat{j}}{\sqrt{2}}$

Solution (A) is the correct answer. Required unit vector is $\frac{(\hat{i} - \hat{j}) \times (\hat{i} + \hat{j})}{|(\hat{i} - \hat{j}) \times (\hat{i} + \hat{j})|} = \frac{2\hat{k}}{2} = \hat{k}$.

Example 21 If $|\vec{a}| = 3$ and $-1 \le k \le 2$, then $|k\vec{a}|$ lies in the interval

(A) [0, 6] (B) [– 3, 6] (C) [3, 6] (D) [1, 2]

Solution (A) is the correct answer. The smallest value of $|k\vec{a}|$ will exist at numerically smallest value of k, i.e., at $k = 0$, which gives $|k\vec{a}| = |k||\vec{a}| = 0 \times 3 = 0$

The numerically greatest value of k is 2 at which $|k\vec{a}| = 6$.

10.3 EXERCISE

Short Answer (S.A.)

1. Find the unit vector in the direction of sum of vectors $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = 2\hat{j} + \hat{k}$.

2. If $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} - 2\hat{k}$, find the unit vector in the direction of

(i) $6\vec{b}$ (ii) $2\vec{a} - \vec{b}$

3. Find a unit vector in the direction of $\overrightarrow{PQ}$, where P and Q have co-ordinates (5, 0, 8) and (3, 3, 2), respectively.

4. If $\vec{a}$ and $\vec{b}$ are the position vectors of A and B, respectively, find the position vector of a point C in BA produced such that BC = 1.5 BA.

5. Using vectors, find the value of k such that the points (k, – 10, 3), (1, –1, 3) and (3, 5, 3) are collinear.

6. A vector $\vec{r}$ is inclined at equal angles to the three axes. If the magnitude of $\vec{r}$ is $2\sqrt{3}$ units, find $\vec{r}$.

7. A vector $\vec{r}$ has magnitude 14 and direction ratios 2, 3, – 6. Find the direction cosines and components of $\vec{r}$, given that $\vec{r}$ makes an acute angle with x-axis.

8. Find a vector of magnitude 6, which is perpendicular to both the vectors $2\hat{i} - \hat{j} + 2\hat{k}$ and $4\hat{i} - \hat{j} + 3\hat{k}$.

9. Find the angle between the vectors $2\hat{i} - \hat{j} + \hat{k}$ and $3\hat{i} + 4\hat{j} - \hat{k}$.

10. If $\vec{a} + \vec{b} + \vec{c} = 0$, show that $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$. Interpret the result geometrically?

11. Find the sine of the angle between the vectors $\vec{a} = 3\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} - 2\hat{j} + 4\hat{k}$.

12. If A, B, C, D are the points with position vectors $\hat{i}+\hat{j}-\hat{k}$, $2\hat{i}-\hat{j}+3\hat{k}$, $2\hat{i}-3\hat{k}$, $3\hat{i}-2\hat{j}+\hat{k}$, respectively, find the projection of $\overrightarrow{AB}$ along $\overrightarrow{CD}$.

13. Using vectors, find the area of the triangle ABC with vertices A(1, 2, 3), B(2, – 1, 4) and C(4, 5, – 1).

14. Using vectors, prove that the parallelogram on the same base and between the same parallels are equal in area.

Long Answer (L.A.)

15. Prove that in any triangle ABC, $\cos A = \frac{b^2+c^2-a^2}{2bc}$, where *a*, *b*, *c* are the magnitudes of the sides opposite to the vertices A, B, C, respectively.

16. If $\vec{a}, \vec{b}, \vec{c}$ determine the vertices of a triangle, show that $\frac{1}{2}\left[\vec{b}\times\vec{c}+\vec{c}\times\vec{a}+\vec{a}\times\vec{b}\right]$ gives the vector area of the triangle. Hence deduce the condition that the three points $\vec{a}, \vec{b}, \vec{c}$ are collinear. Also find the unit vector normal to the plane of the triangle.

17. Show that area of the parallelogram whose diagonals are given by $\vec{a}$ and $\vec{b}$ is $\frac{|\vec{a}\times\vec{b}|}{2}$. Also find the area of the parallelogram whose diagonals are $2\hat{i}-\hat{j}+\hat{k}$ and $\hat{i}+3\hat{j}-\hat{k}$.

18. If $\vec{a} = \hat{i}+\hat{j}+\hat{k}$ and $\vec{b} = \hat{j}-\hat{k}$, find a vector $\vec{c}$ such that $\vec{a}\times\vec{c}=\vec{b}$ and $\vec{a}.\vec{c}=3$.

Objective Type Questions

Choose the correct answer from the given four options in each of the Exercises from 19 to 33 (M.C.Q)

19. The vector in the direction of the vector $\hat{i}-2\hat{j}+2\hat{k}$ that has magnitude 9 is

(A) $\hat{i}-2\hat{j}+2\hat{k}$ (B) $\frac{\hat{i}-2\hat{j}+2\hat{k}}{3}$

(C) $3(\hat{i}-2\hat{j}+2\hat{k})$ (D) $9(\hat{i}-2\hat{j}+2\hat{k})$

20. The position vector of the point which divides the join of points $2\vec{a}-3\vec{b}$ and $\vec{a}+\vec{b}$ in the ratio 3 : 1 is

(A) $\dfrac{3\vec{a}-2\vec{b}}{2}$ (B) $\dfrac{7\vec{a}-8\vec{b}}{4}$ (C) $\dfrac{3\vec{a}}{4}$ (D) $\dfrac{5\vec{a}}{4}$

21. The vector having initial and terminal points as (2, 5, 0) and (–3, 7, 4), respectively is

(A) $-\hat{i}+12\hat{j}+4\hat{k}$ (B) $5\hat{i}+2\hat{j}-4\hat{k}$

(C) $-5\hat{i}+2\hat{j}+4\hat{k}$ (D) $\hat{i}+\hat{j}+\hat{k}$

22. The angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes $\sqrt{3}$ and 4, respectively, and $\vec{a}.\vec{b}=2\sqrt{3}$ is

(A) $\dfrac{\pi}{6}$ (B) $\dfrac{\pi}{3}$ (C) $\dfrac{\pi}{2}$ (D) $\dfrac{5\pi}{2}$

23. Find the value of λ such that the vectors $\vec{a}=2\hat{i}+\lambda\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}+3\hat{k}$ are orthogonal

(A) 0 (B) 1 (C) $\dfrac{3}{2}$ (D) $-\dfrac{5}{2}$

24. The value of λ for which the vectors $3\hat{i}-6\hat{j}+\hat{k}$ and $2\hat{i}-4\hat{j}+\lambda\hat{k}$ are parallel is

(A) $\dfrac{2}{3}$ (B) $\dfrac{3}{2}$ (C) $\dfrac{5}{2}$ (D) $\dfrac{2}{5}$

25. The vectors from origin to the points A and B are $\vec{a}=2\hat{i}-3\hat{j}+2\hat{k}$ and $\vec{b}=2\hat{i}+3\hat{j}+\hat{k}$, respectively, then the area of triangle OAB is

(A) 340 (B) $\sqrt{25}$ (C) $\sqrt{229}$ (D) $\dfrac{1}{2}\sqrt{229}$

26. For any vector $\vec{a}$, the value of $(\vec{a}\times\hat{i})^2+(\vec{a}\times\hat{j})^2+(\vec{a}\times\hat{k})^2$ is equal to

(A) $\vec{a}^2$ (B) $3\vec{a}^2$ (C) $4\vec{a}^2$ (D) $2\vec{a}^2$

27. If $|\vec{a}|$ = 10, $|\vec{b}|$ = 2 and $\vec{a}.\vec{b}=12$, then value of $|\vec{a}\times\vec{b}|$ is

(A) 5 (B) 10 (C) 14 (D) 16

28. The vectors $\lambda\hat{i}+\hat{j}+2\hat{k}$, $\hat{i}+\lambda\hat{j}-\hat{k}$ and $2\hat{i}-\hat{j}+\lambda\hat{k}$ are coplanar if

(A) $\lambda = -2$ (B) $\lambda = 0$ (C) $\lambda = 1$ (D) $\lambda = -1$

29. If $\vec{a},\vec{b},\vec{c}$ are unit vectors such that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, then the value of $\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}$ is

(A) 1 (B) 3 (C) $-\frac{3}{2}$ (D) None of these

30. Projection vector of $\vec{a}$ on $\vec{b}$ is

(A) $\left(\frac{\vec{a}.\vec{b}}{|\vec{b}|^2}\right)\vec{b}$ (B) $\frac{\vec{a}.\vec{b}}{|\vec{b}|}$ (C) $\frac{\vec{a}.\vec{b}}{|\vec{a}|}$ (D) $\left(\frac{\vec{a}.\vec{b}}{|\vec{a}|^2}\right)\hat{b}$

31. If $\vec{a},\vec{b},\vec{c}$ are three vectors such that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$ and $|\vec{a}|=2$, $|\vec{b}|=3$, $|\vec{c}|=5$, then value of $\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}$ is

(A) 0 (B) 1 (C) – 19 (D) 38

32. If $|\vec{a}|=4$ and $-3\le\lambda\le 2$, then the range of $|\lambda\vec{a}|$ is

(A) [0, 8] (B) [– 12, 8] (C) [0, 12] (D) [8, 12]

33. The number of vectors of unit length perpendicular to the vectors $\vec{a}=2\hat{i}+\hat{j}+2\hat{k}$ and $\vec{b}=\hat{j}+\hat{k}$ is

(A) one (B) two (C) three (D) infinite

Fill in the blanks in each of the Exercises from 34 to 40.

34. The vector $\vec{a}+\vec{b}$ bisects the angle between the non-collinear vectors $\vec{a}$ and $\vec{b}$ if ________

35. If $\vec{r}.\vec{a}=0, \vec{r}.\vec{b}=0$, and $\vec{r}.\vec{c}=0$ for some non-zero vector $\vec{r}$, then the value of $\vec{a}.(\vec{b}\times\vec{c})$ is ________

36. The vectors $\vec{a}=3i-2j+2\hat{k}$ and $\vec{b}=-\hat{i}-\widehat{2k}$ are the adjacent sides of a parallelogram. The acute angle between its diagonals is ________.

37. The values of k for which $|k\vec{a}|<|\vec{a}|$ and $k\vec{a}+\frac{1}{2}\vec{a}$ is parallel to $\vec{a}$ holds true are _______.

38. The value of the expression $|\vec{a}\times\vec{b}|^2+(\vec{a}.\vec{b})^2$ is _______.

39. If $|\vec{a}\times\vec{b}|^2+|\vec{a}.\vec{b}|^2 = 144$ and $|\vec{a}|=4$, then $|\vec{b}|$ is equal to _______.

40. If $\vec{a}$ is any non-zero vector, then $(\vec{a}.\hat{i})\hat{i}+(\vec{a}.\hat{j})\hat{j}+(\vec{a}.\hat{k})\hat{k}$ equals _______.

State **True** or **False** in each of the following Exercises.

41. If $|\vec{a}|=|\vec{b}|$, then necessarily it implies $\vec{a}=\pm\vec{b}$.

42. Position vector of a point P is a vector whose initial point is origin.

43. If $|\vec{a}+\vec{b}|=|\vec{a}-\vec{b}|$, then the vectors $\vec{a}$ and $\vec{b}$ are orthogonal.

44. The formula $(\vec{a}+\vec{b})^2=\vec{a}^2+\vec{b}^2+2\vec{a}\times\vec{b}$ is valid for non-zero vectors $\vec{a}$ and $\vec{b}$.

45. If $\vec{a}$ and $\vec{b}$ are adjacent sides of a rhombus, then $\vec{a}\cdot\vec{b} = 0$.